Fluctuation dissipation theorems and irreversible thermodynamics

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We investigate the statistics of fluctuations in macroscopic systems described by thermodynamics. We begin by reviewing fluctuations in the context of linear irreversible thermodynamics and show that a more direct characterization of the fluctuations is possible, if velocity fluctuations are explicitly included in the second variation of the entropy, $\delta^2 S$, about the equilibrium state. A similar procedure is then applied to what is the main goal of this paper: elucidating the nature of fluctuations in hyperbolic macroscopic systems, where signals have a finite transmission velocity. We find that, once again, velocity fluctuations have to be explicitly included, which takes us outside of extended irreversible thermodynamics as it is often defined. We find the explicit form of the fluctuation-dissipation theorem in this case, and determine the statistics of the stochastic variables in terms of the quantities appearing in the deterministic dynamics. The fluctuating theory is then reformulated in order to elucidate the relationship between the extended theory and linear irreversible thermodynamics. This has the effect of bringing out the general structure more clearly: the real, frequency-independent transport coefficients of linear irreversible thermodynamics are replaced by their complex, frequency-dependent counterparts in the extended theory.

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I. INTRODUCTION

The problem of the description of equilibrium and nonequilibrium fluctuations in macroscopic systems is one of the central aspects of both thermodynamic and microscopic theories of irreversible processes. This problem has been studied by many researchers since the 1960's [1]. From the macroscopic viewpoint, the nonequilibrium case has been investigated within the framework of several theories among which we mention in particular extended irreversible thermodynamic theory [2]. The starting point of extended thermodynamics is the generalization of the Gibbs relation for the nonequilibrium entropy, which is used to determine the second moments of the physical fields under the assumption that the probability of the fluctuations is given by the Einstein relation [3]. Much effort has also been expended within the theory of stochastic processes to obtain the mesoscopic basis of the macroscopic theories. The stochastic formulation, dating back to Onsager and Machlup [4], is based on stochastic processes that are stationary, Gaussian and Markovian. The starting assumption is that the system is well described by a set of macroscopic variables $a(\mathbf{r},t)$, a subset of which will not be conserved, and that the state of the system is well defined at each position and time in terms of such set of properties. They are considered to take on continuous values and vary continuously in space and time. The behavior of variables $a_b(\mathbf{r},t)$, where b labels the variables, is then approximated by the Langevin-type equations

$$\frac{\partial a_b(\mathbf{r},t)}{\partial t} = -\sum_c \int d\mathbf{r}'\, G_{bc}(\mathbf{r},\mathbf{r}')\, a_c(\mathbf{r}',t) + \tilde{f}_b(\mathbf{r},t), \quad (1)$$

where the first term on the right-hand side is a result of the linearization of the macroscopic equation about the stationary state and $\tilde{f}_b(\mathbf{r},t)$ is a stochastic term that represents fluctuations in the system.

Such a theory implies a general fluctuation dissipation theorem (FDT), which is valid even in systems without local equilibrium [5]. This theorem provides us with, in principle, a tool to determine the statistics of $\tilde{f}_b(\mathbf{r},t)$. The statistical properties of the physical fields arise from this fluctuating stochastic term that is assumed to be Gaussian and Markovian (white). The FDT is valid even without any thermodynamics because it is a consequence of the Langevin dynamics alone. The phenomenological theory of fluctuations obtained from such a theorem is well posed in the sense that its extension to nonequilibrium states, arbitrarily far from equilibrium is immediately valid and no further physical arguments are needed. However, the FDT simply relates the statistics of the $\tilde{f}_b$ to quantities appearing in the macroscopic theory (properties of G_{bc}), and thus a theory of nonequilibrium thermodynamics is required to make use of the FDT in practice.

Our concern in this paper will be the phenomenological theory of fluctuations in nonequilibrium macroscopic systems in which thermodynamic inertia plays an important dynamical role. The inclusion of thermodynamic inertia in the description leads us into the domain of hyperbolic phenomena, i.e., a set of phenomena in which signals have a finite transmission velocity. Examples of these are diffusive and dissipative transport phenomena in viscoelastic fluids, second sound in heat transmission, etc. Extended irreversible thermodynamics (EIT) is the natural thermodynamic framework for these kind of phenomena [2], and we shall show that the approach used to determine the FDT in systems without inertia can be extended to find the FDT in EIT.

The outline of the paper is as follows. We begin in Sec. II with a derivation of the FDT and then go on in Sec. III to show how it can be used to specify the statistics of the stochastic term in linear irreversible thermodynamics (LIT). We begin by closely following the approach adopted by Fox and Uhlenbeck [6] (which is in essence that of Landau and Lifshitz [7]), but we will use a more direct method than they did, which we will argue is less ambiguous. In Sec. IV we carry over the same methods to EIT and find the resulting FDT. In Sec. V we will describe an alternative way of presenting the results of Sec. IV, which more closely resembles the FDT in LIT. The stochastic terms are now, however, no longer δ-function correlated, but exponentially correlated in time. An Appendix gives some technical details that were omitted from the main part of the paper.

II. FLUCTUATION-DISSIPATION THEOREM AND IRREVERSIBLE THERMODYNAMICS

In this section we will derive the FDT for equations of the form (1). As we have stressed in Sec. I, the theorem follows only from the form of the stochastic dynamics and so can be proved independently of any thermodynamic description that we will later use.

In this paper we will frequently adopt an abbreviated form where the continuous labels $\mathbf{r},\mathbf{r}'$ are replaced by the discrete labels j,k for convenience and where the summation convention is assumed. So, for instance, equations of the type (1) are written in the form

$$\dot{a}_b^j(t) + G_{bc}^{jk} a_c^k(t) = \tilde{f}_b^j(t). \tag{2}$$

The stochastic term $\tilde{f}_b^j(t)$ is taken to have a Gaussian distribution with mean zero and correlator

$$\langle \tilde{f}_b^j(t) \tilde{f}_c^k(t') \rangle = 2 Q_{bc}^{jk} \delta(t-t'). \tag{3}$$

It is clear from Eq. (3) that the matrix Q, viewed in the combined (j,b) space, is real, symmetric, and positive semidefinite. The FDT constitutes the link between the stochastic formalism discussed above and thermodynamics. The generality of this theorem is not widely appreciated; it includes hyperbolic phenomena in spatially extended systems—which is of our interest here—and systems out of local equilibrium, as has been remarked by Eyink *et al.* [5]. As mentioned above, the only assumption required is that the variables are described by some generalized Langevin equation of the kind (2). The variables $a_c^k(t)$ represent linear deviations from the stationary state. Since, by Eq. (2), they are linearly related to $\tilde{f}_b^j$, which are Gaussian random variables, they are themselves Gaussian random variables. For an aged system, that is one where the initial conditions were set in the infinitely distant past, $\langle a_e^l \rangle = 0$ and the probability distribution of the a is stationary:

$$P_S(a) = \mathcal{N} \exp\left\{ -\frac{1}{2} a_b^j E_{bc}^{jk} a_c^k \right\}, \tag{4}$$

where $\mathcal{N}$ is a normalization constant and where the subscript S is a reminder that this is a time-independent probability density function. The E_{bc}^{jk} matrix is, for the moment, undetermined, but we note that it follows from Eq. (4) that

$$\langle a_e^l a_f^m \rangle_S = (E^{-1})_{ef}^{lm}. \tag{5}$$

Since the stochastic process defined by Eqs. (2) and (3) only depends on Q_{ab}^{ij} and G_{ab}^{ij}, E_{ab}^{ij} must be related to these two matrices. This relationship is the FD theorem. To determine it, we solve Eq. (2) and from the solution determine the correlation function in the stationary state to be [8]

$$\langle a_e^l(0) a_f^m(0) \rangle_S = 2 \int_0^\infty d\rho \; e^{-\rho G_{eb}^{lj}} Q_{bc}^{jk} e^{-\rho G_{fc}^{mk}}. \tag{6}$$

Since the initial conditions were set in the infinitely distant past, and the a's are both evaluated at $t=0$, the correlation function on the left-hand side of Eq. (6) is equal to the one on the left-hand side of Eq. (5) and so equal to $(E^{-1})_{ef}^{lm}$. Performing the integral in Eq. (6) gives the FD theorem

$$2 Q_{ab}^{ij} = G_{ac}^{ik} (E^{-1})_{cb}^{kj} + (E^{-1})_{ac}^{ik} G_{cb}^{Tkj}, \tag{7}$$

where T denotes transpose. We stress again that no condition of time irreversibility or detailed balance was required to derive Eq. (7); it is simply a consequence of the large time behavior of a system described by Eqs. (2) and (3).

III. FLUCTUATIONAL DYNAMICS FROM LINEAR IRREVERSIBLE THERMODYNAMICS

In this section we will review the theory of hydrodynamic fluctuations in LIT. Our purpose is not only to provide an introduction that serves as background to the corresponding theory in EIT, but also to clarify some points of confusion concerning this problem that exist in the literature.

We begin from the well-known balance equations for mass, linear momentum, and energy

$$\rho \frac{Dv}{Dt} = \frac{\partial v_\mu}{\partial x_\mu}, \tag{8}$$

$$\rho \frac{Dv_\mu}{Dt} = -\frac{\partial P_{\mu\nu}}{\partial x_\nu} + \rho F_\mu, \tag{9}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial q_\mu}{\partial x_\mu} - P_{\mu\nu} V_{\nu\mu}, \tag{10}$$

with the constitutive relations

$$\mathring{\tau}_{\mu\nu} = -2\mu \mathring{V}_{\mu\nu}, \quad q_\mu = -\lambda \frac{\partial T}{\partial x_\mu}, \quad \tau = -\zeta \frac{\partial v_\mu}{\partial x_\mu}. \tag{11}$$

We use the convention that $\mu,\nu = 1,2,3$. In Eqs. (8)–(10), D/Dt is the material derivative, v is the volume per unit mass, u is the internal energy per unit mass, q_μ is the heat flux, ρ is the mass density, F_μ the external body forces per unit mass, v_μ the barycentric velocity, $V_{\mu\nu}$ the symmetric part of the gradient velocity, and $P_{\mu\nu}$ the pressure tensor that is written as

$$P_{\mu\nu}=(p+\tau)\delta_{\mu\nu}+\mathring{\tau}_{\mu\nu},$$

p being the thermodynamic pressure and $\tau_{\mu\nu}$ the stress viscous tensor. Repeated indices are summed and a circle over a tensor symbol indicates that it is traceless. In Eq. (11) $q_\mu, \mathring{\tau}_{\mu\nu}, \tau$ represent the heat flux, the traceless stress tensor, and its trace, respectively. In addition λ, ζ, and μ are the thermal conductivity, the bulk viscosity, and the shear viscosity, respectively.

We now wish to express the equations governing the fluctuations about the equilibrium state in the Langevin form (2) where the stochastic terms have a correlator of the form (3). The process therefore consists of two stages:

(i) A linearization of Eqs. (8)–(11) about the equilibrium state in order to make contact with the Langevin equations (2).

(ii) The use of an Einstein relation, for the probability of fluctuations about the equilibrium state, to determine the matrix E^{jk}_{bc} by comparison with Eq. (4). Use of the FDT (7) then allows Q^{jk}_{bc} to be determined.

The linearization about the equilibrium state defined by $(v,v_\mu,T)=(v_0,0,T_0)$ has been clearly discussed by Fox and Uhlenbeck [6], and we will simply summarize the essential points here. We will denote volume and temperature fluctuations by v_1 and T_1, respectively, $v=v_0+v_1$ and $T=T_0+T_1$, but will use the same notation for the velocity and the velocity fluctuations, since no confusion should arise. The result of this linearization gives

$$\dot{a}^j_b(t)+G^{jk}_{bc}a^k_c(t)=0; \quad b,c=1,\ldots,5, \tag{12}$$

where j and k represent the spatial degrees of freedom as in Sec. I. We will not give the explicit form for the G^{jk}_{bc} here; we will limit our discussion to giving the relationship between the fluctuating quantities $a^j_b(t)$ and the variables appearing in Eqs. (8)–(11). The set of variables $\{v,v_\mu,T\}$ that form LIT comprise five independent components. The $a_b(\mathbf{r},t)$ are simply scaled versions of these variables.

$$a_1=-\rho_0^{3/2}v_1, \quad a_{\mu+1}=\left(\frac{\rho_0}{A}\right)^{1/2}(v_1)_\mu, \quad a_5=\left(\frac{\rho_0 C}{T_0 A}\right)^{1/2}T_1, \tag{13}$$

where A and C are quantities defined solely in terms of the fluid in equilibrium.

$$A\equiv\left(\frac{\partial p}{\partial\rho}\right)_T, \quad C\equiv\left(\frac{\partial u}{\partial T}\right)_v. \tag{14}$$

In this paper all partial derivatives with a subscript outside of the brackets, as in Eq. (14), will denote equilibrium quantities with the subscript denoting the quantity that is kept fixed during the variation. We will also sometimes write $(\partial_x y)_z$ for $(\partial y/\partial x)_z$.

We now move on to the second stage by adding stochastic terms to the right-hand side of Eq. (12) to represent the effects of all of the other degrees of freedom in the fluid that have been omitted in the description of the fluid given by Eqs. (8)–(11). To determine E^{jk}_{bc}, Fox and Uhlenbeck proceeded indirectly: the Einstein relation

$$P_S(a)\sim\exp\{\delta^2S/2k_B\}, \tag{15}$$

when taken in conjunction with Eq. (4), gives to quadratic order

$$S(a)=S_{\mathrm{eq}}-\frac{1}{2}k_B a^j_b E^{jk}_{bc}a^k_c. \tag{16}$$

From this it follows that

$$\frac{dS}{dt}=-k_B\dot{a}^j_b E^{jk}_{bc}a^k_c=+k_B a^j_b(G^T E)^{jk}_{bc}a^k_c, \tag{17}$$

where Eq. (12) has been used in the final step.

On the other hand, starting from thermodynamics and the balance equations they showed that

$$\frac{dS}{dt}=k_B a^j_b\left(\frac{A}{k_B T_0}S^{jk}_{bc}\right)a^k_c, \tag{18}$$

where S^{jk}_{bc} is the symmetric part of the matrix G^{jk}_{bc}. Note that this is a symmetry in the combined (j,b) space as in Sec. I, i.e.,

$$G^{jk}_{bc}=S^{jk}_{bc}+A^{jk}_{bc}, \tag{19}$$

where

$$S^{jk}_{bc}=S^{kj}_{cb} \quad \text{and} \quad A^{jk}_{bc}=-A^{kj}_{cb}. \tag{20}$$

Comparing Eqs. (17) and (18) they then deduce that

$$E^{jk}_{bc}=\frac{A}{k_B T_0}\delta_{jk}\delta_{bc}. \tag{21}$$

This method of arriving at Eq. (21) is, in our opinion, rather indirect and requires additional assumptions [for example, if we assume that E^{jk}_{bc} is proportional to the unit matrix, then Eq. (21) follows from Eqs. (17) and (18)]. We would rather determine E^{jk}_{bc} by direct comparison with the thermodynamic expression for δ^2S, rather than with dS/dt. This provides a direct determination of E^{jk}_{bc}, but it also clarifies some points of confusion surrounding the role of velocity in LIT.

Much of the early literature on LIT seemed not to directly address the contribution of the velocity fluctuations to δ^2S. For example, in the classic text by Callen [9] we find only the following correlation functions

$$\langle u_1^2\rangle=-\frac{k_B}{M}\left(\frac{\partial u}{\partial(1/T)}\right)_{P/T}, \tag{22}$$

$$\langle u_1 v_1 \rangle = -\frac{k_B}{M}\left(\frac{\partial v}{\partial(1/T)}\right)_{P/T}, \tag{23}$$

$$\langle v_1^2 \rangle = -\frac{k_B}{M}\left(\frac{\partial v}{\partial(P/T)}\right)_{1/T}, \tag{24}$$

where M is the total mass of the fluid. The Gaussian probability function that gives rise to these correlations is easily constructed by inverting the matrix that has Eqs. (22)–(24) as its entries (c.f. (4) and (5)). This can be achieved by using the chain rule for partial derivatives that transforms between the set of independent extensive variables u and v and the set of corresponding intensive variables $1/T$ and p/T. One finds, restoring the spatial dependence to u_1 and v_1,

$$P_S(v_1,u_1) \sim \exp\left\{\frac{M}{2k_B V}[(\partial_v\{T^{-1}p\})_u v_1^j v_1^j + 2(\partial_v\{T^{-1}\})_u v_1^j u_1^j + (\partial_u\{T^{-1}\})_v u_1^j u_1^j]\right\}, \tag{25}$$

where V is the total volume of the fluid. The expression (25) is not in a form that is immediately useful to us, since the independent variables v and u are used instead of v and T used by Fox and Uhlenbeck. The transformation from $\{v_1,u_1\}$ to $\{v_1,T_1\}$ is given by Eq. (A3) in the Appendix. Using the results (A6) and (A7), also from the Appendix, one finds that

$$(\partial_v\{T^{-1}p\})_u v_1^j v_1^j + 2(\partial_v\{T^{-1}\})_u v_1^j u_1^j + (\partial_u\{T^{-1}\})_v u_1^j u_1^j = -\frac{A\rho_0^2}{T_0} v_1^j v_1^j - \frac{C}{T_0^2} T_1^j T_1^j. \tag{26}$$

Substituting Eq. (26) into Eq. (25), and using the rescaled variables (13), we obtain

$$P_S(a) \sim \exp\left\{\frac{A}{2k_B T_0}[-a_1^j a_1^j - a_5^j a_5^j]\right\}, \tag{27}$$

where A is given by Eq. (14). Since, E_{bc}^{jk} can be read off by comparing Eq. (27) with Eq. (4), we can determine the Q_{bc}^{jk} by using the FD theorem (7). However, there is a problem with the relation (27)—it does not depend on $a_{\mu+1}^j$, the velocity fluctuations. Therefore, although E_{bc}^{jk} is diagonal, some of the entries are zero and so it has no inverse. Consequently, $\langle a_e^l a_f^m \rangle$, $(e,f=2,3,4)$, is formally infinite.

This shows the necessity of including the velocity fluctuations in $\delta^2 S$. The need to do this was pointed out by Glansdorff and Prigogine [10] and subsequently clarified by several authors [11–14], although it is still not widely appreciated. Briefly, if the extensive variables are u and v and the corresponding intensive variables are $1/T$ and p/T, then from Eqs. (15) and (25)

$$\delta^2 s = \delta u\,\delta(1/T) + \delta v\,\delta(p/T), \tag{28}$$

where $\delta u = u_1$ and $\delta v = v_1$, and where as usual the use of lower case s implies that we are considering the entropy per unit mass. Since the E matrix obtained using this starting point is only 2×2, we need to include velocity fluctuations to get the full matrix. In this case we should work with the extensive variables u, v, and v_μ and the corresponding intensive variables $1/T$, p/T, and $-v_\mu/T$. The expression (28) for $\delta^2 s$ should be replaced by

$$\delta^2 s = \delta u\,\delta(1/T) + \delta v\,\delta(p/T) + \delta v_\mu\,\delta(-v_\mu/T). \tag{29}$$

Since the velocity fluctuations are simply the velocity to this order, we may write the last term, again to this order, as $-v_\mu v_\mu/T_0$. Therefore, an extra term $-Mv_\mu v_\mu/2k_B T_0$ has to be added on to the exponent in Eq. (25). After rescaling according to Eq. (13), the factor $-a_1^j a_1^j - a_5^j a_5^j$ is replaced by simply $-a_b^j a_b^j$, and so E_{bc}^{jk} is given by Eq. (21), as required. However, identifying E_{bc}^{jk} directly through $\delta^2 s$, rather than indirectly through ds/dt, is more satisfactory since no other assumptions about the form of E_{bc}^{jk} are required.

Substituting Eq. (21) into Eq. (7) gives

$$Q_{bc}^{jk} = \frac{k_B T_0}{A} S_{bc}^{jk}. \tag{30}$$

For the purposes of comparison with our later results we give the explicit form for S_{bc}^{jk} as derived in Ref. [6]:

$$S_{\mu+1,\nu+1}(\mathbf{r},\mathbf{r}') = \frac{1}{\rho_0}[2\mu X_{\mu\rho\nu\sigma} + \zeta\,\delta_{\mu\rho}\delta_{\nu\sigma}] \times \frac{\partial^2}{\partial x_\rho\,\partial x'_\sigma}\,\delta(\mathbf{r}-\mathbf{r}'), \tag{31}$$

$$S_{55}(\mathbf{r},\mathbf{r}') = \frac{1}{\rho_0 C}\lambda\,\delta_{\mu\nu}\frac{\partial^2}{\partial x_\mu\,\partial x'_\nu}\,\delta(\mathbf{r}-\mathbf{r}'), \tag{32}$$

with all other $S_{bc}(\mathbf{r},\mathbf{r}')$, including $S_{11}(\mathbf{r},\mathbf{r}')$, equal to zero. The tensor $X_{\mu\nu\rho\sigma}$ is defined by

$$X_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \frac{2}{3}\delta_{\mu\nu}\delta_{\rho\sigma}\right). \tag{33}$$

In Eqs. (31) and (32) the continuum limit has been taken so that the discrete spatial variables j,k have been replaced by $\mathbf{r}$ and $\mathbf{r}'$. Note the presence of spatial derivatives in Eqs. (31) and (32), and therefore through Eq. (30), in $Q_{bc}(\mathbf{r},\mathbf{r}')$. These can be eliminated if, instead of working with the stochastic terms $\tilde{f}_{\mu+1}(\mathbf{r},t)$ and $\tilde{f}_5(\mathbf{r},t)$, one introduces new quantities $\tilde{s}_{\mu\nu}$ and $\tilde{g}_\mu$ by [6]

$$\tilde{f}_{\mu+1}(\mathbf{r},t) = (\rho_0 A)^{-1/2}\frac{\partial}{\partial x_\nu}\tilde{s}_{\mu\nu}(\mathbf{r},t), \tag{34}$$

$$\tilde{f}_5(\mathbf{r},t) = (\rho_0 T_0 A C)^{-1/2}\frac{\partial}{\partial x_\mu}\tilde{g}_\mu(\mathbf{r},t). \tag{35}$$

From Eqs. (3), (30), and the above definitions, one finds the FDT in the form

$$\langle \tilde{s}_{\mu\nu}(\mathbf{r},t)\tilde{s}_{\rho\sigma}(\mathbf{r}',t')\rangle = 2k_B T_0[2\mu X_{\mu\nu\rho\sigma} + \zeta\delta_{\mu\nu}\delta_{\rho\sigma}] \times \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \quad (36)$$

$$\langle \tilde{g}_{\mu}(\mathbf{r},t)\tilde{g}_{\nu}(\mathbf{r}',t')\rangle = 2k_B T_0^2 \lambda\,\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \quad (37)$$

with all other contributions vanishing. In Sec. V we will show how these results are generalized when hyperbolic phenomena are taken into account.

In this section we have investigated the Langevin description of a nonequilibrium fluid obtained by adding stochastic terms to the linearized hydrodynamic equations around the equilibrium state. Since this description is not obtained through a coarse grained process the stochastic properties of the fluctuating terms represented by the Q_{bc}^{ij} matrix in Eq. (3) must be determined by use of the FDT (7). Also since the stochastic terms $f_b(\mathbf{r},t)$ are Gaussian with zero mean, they are completely characterized by this matrix. The well-known relation (30) shows that Q_{bc} is essentially the symmetric part of G_{bc}, which may be obtained from the linearized hydrodynamics. In the next section we derive an analogous relation for EIT.

IV. FLUCTUATIONAL DYNAMICS FROM EXTENDED IRREVERSIBLE THERMODYNAMICS

In this section we will consider phenomena in which the frequency of the perturbations becomes comparable to the inverse of the relaxation times of the dissipative fluxes. We also assume a finite velocity of transmission of signals in the system. It is well known that in such nonequilibrium conditions the dissipative fluxes are not uniquely determined by the spatial gradients of the hydrodynamic densities. One way to construct a thermodynamics for these kinds of nonequilibrium states is to change the usual thermodynamic role of the fluxes and to elevate them to the same status as the independent slow variables of the system. This gives rise to EIT [2]. The existence of a generalized nonequilibrium entropy depending on the extended set of variables including the dissipative fluxes is assumed. By further assuming that this entropy function is differentiable, it is possible to get a closed set of dynamic equations made up of the well-known balance equations for mass, linear momentum, and energy (8)–(10), and the constitutive equations for the new independent variables [2], whose simplest form that allows us to deal with hyperbolic phenomena is

$$\tau_2 \frac{\partial \mathring{\tau}_{\mu\nu}}{\partial t} + \mathring{\tau}_{\mu\nu} = -2\mu \mathring{V}_{\mu\nu}, \quad (38)$$

$$\tau_1 \frac{\partial q_\mu}{\partial t} + q_\mu = -\lambda \frac{\partial T}{\partial x_\mu}, \quad (39)$$

$$\tau_0 \frac{\partial \tau}{\partial t} + \tau = -\zeta \frac{\partial v_\mu}{\partial x_\mu}, \quad (40)$$

where the τ_i, $i=0,1,2$, are the relaxation times of the various fluxes.

This way of proceeding, and the thermodynamics implied by the hypothesis of the enlargement of the thermodynamic variables space, is understood as an extension of linear irreversible thermodynamics to describe far from equilibrium phenomena [15]. Its microscopic basis appears to be found in the kinetic theory of Boltzmann for a dilute monoatomic gas through the 13 moments method of solution given by Grad [16], in generalizations of the moments method [17] or in information theory [18]. The model defined by Eqs. (38)–(40), has been used successfully in the study of hyperbolic phenomena in viscoelastic fluids, heat conduction, diffusion, etc. [2].

The linearization is carried out in exactly the same way as in Sec. II, except that here the equilibrium state is defined by $(v,v_\mu,T,\mathring{\tau}_{\mu\nu},q_\mu,\tau)=(v_0,0,T_0,0,0,0)$. As in the case of velocity in LIT, we will use the same notation for the extra variables introduced in EIT and the fluctuations about them, since no confusion should arise. The result of this linearization gives Eq. (12), but with $b,c=1,\ldots,14$. The form of G_{bc}^{jk} is given in the Appendix along with the details of the derivation. Here we once again concentrate on the relationship between the fluctuating quantities $a_b^j(t)$ and the variables appearing in Eqs. (8)–(10) and (38)–(40). Since $\mathring{\tau}_{\mu\nu}$ has five independent components, it is clear that the set of variables $\{v,v_\mu,T,\mathring{\tau}_{\mu\nu},q_\mu,\tau\}$ that form EIT—with the addition of velocity as an independent thermodynamic variable—comprise of 14 independent components. The scaled versions of the first five of these variables are given by Eq. (13); the other nine are given by

$$\mathring{a}_{\mu\nu} = \left(\frac{\tau_2}{\mu A}\right)^{1/2}\mathring{\tau}_{\mu\nu}, \quad a_{\mu+10} = \left(\frac{\tau_1}{T_0\lambda A}\right)^{1/2} q_\mu,$$
$$a_{14} = \left(\frac{\tau_0}{\zeta A}\right)^{1/2}\tau. \quad (41)$$

These scalings are chosen so that all the a_b have the same dimension (of the square root of density). The only slight subtlety comes in the specification of the five independent degrees of freedom corresponding to $\mathring{a}_{\mu\nu}(\mathbf{r},t)$. There is no single, natural mapping on to $\{a_b(\mathbf{r},t)|b=6,\ldots,10\}$. For example one could take a_6, a_7, and a_8 to be the three off-diagonal entries $\mathring{a}_{\mu\nu}$ with $\mu<\nu$, and $a_9=\mathring{a}_{11}, a_{10}=\mathring{a}_{22}$. For most of what follows, this choice will not be of any consequence, and we will frequently write $\{\mathring{a}_{\mu\nu}\}$ for $\{a_b|b=6,\ldots,10\}$.

To determine the statistics of the fluctuations we use one of the central results of EIT, namely the probability of fluctuations about the equilibrium state [2]

$$P_S(v_1,u_1,\mathring{\tau}_{\mu\nu},q_\mu,\tau) \sim \exp\left\{\frac{M}{2k_B V}\left[(\partial_v\{T^{-1}p\})_u v_1^j v_1^j + 2(\partial_v\{T^{-1}\})_u v_1^j u_1^j + (\partial_u\{T^{-1}\})_v u_1^j u_1^j - \frac{v\tau_2}{2\mu T_0}\mathring{\tau}^{\,j}_{\mu\nu}\mathring{\tau}^{\,j}_{\nu\mu} - \frac{v\tau_1}{\lambda T_0^2} q_\mu^j q_\mu^j - \frac{v\tau_0}{\zeta T_0}\tau^j\tau^j\right]\right\}. \quad (42)$$

Substituting Eq. (26) into Eq. (25), and using the rescaled variables (13) and (41), we obtain

$$P_S(a) \sim \exp\left\{ \frac{A}{2k_BT_0}\left[-a_1^j a_1^j - a_5^j a_5^j - \frac{1}{2}\mathring{a}^j_{\mu\nu}\mathring{a}^j_{\nu\mu} - a^j_{\mu+10}a^j_{\mu+10} - a^j_{14}a^j_{14}\right]\right\}. \quad (43)$$

The expression (43) suffers from the same problem as Eq. (27) did in LIT. The identification of an E^{jk}_{bc} would lead to inconsistencies because of the omission of the velocity terms in the expression for $\delta^2 s$. If we include them in the way described in Sec. II, we find

$$P_S(a) \sim \exp\left\{ \frac{A}{2k_BT_0}[-a_b^j a_b^j]\right\}. \quad (44)$$

Comparing Eq. (44) with Eq. (4), once again gives Eq. (21). The use of the fluctuation dissipation theorem (7) then gives Eq. (30). From Eq. (A24) we see that, unlike LIT, the S^{jk}_{bc} are diagonal in the spatial variables j,k in EIT, so that $S^{jk}_{bc} \equiv S_{bc}\delta_{jk} \to S_{bc}\delta(\mathbf{r}-\mathbf{r}')$ in the continuum limit. We can therefore write Eq. (30), with the spatial degrees of freedom displayed explicitly as

$$Q_{bc}(\mathbf{r},\mathbf{r}') = \frac{k_BT_0}{A}S_{bc}\delta(\mathbf{r}-\mathbf{r}'), \quad (45)$$

whenever the velocity is included in the thermodynamic variables space.

The vanishing of the Q^{jk}_{bc} matrix for $b,c=1,\ldots,5$ implies that there is no stochastic term in the Eqs. (2) for $b=1,\ldots,5$: $\tilde{f}_1$, $\tilde{f}_{\mu+1}$, and $\tilde{f}_5$ vanish. The specific form of the correlators of the stochastic terms for the other equations may be obtained from Eqs. (3), (45), and (A24), and are

$$\langle \tilde{\mathring{f}}_{\mu\nu}(\mathbf{r},t)\tilde{\mathring{f}}_{\rho\sigma}(\mathbf{r}',t')\rangle = \frac{2k_BT_0}{A\tau_2}X_{\mu\nu\rho\sigma}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'),$$

$$\langle \tilde{f}_{\mu+10}(\mathbf{r},t)\tilde{f}_{\nu+10}(\mathbf{r}',t')\rangle = \frac{2k_BT_0}{A\tau_1}\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'),$$

$$\langle \tilde{f}_{14}(\mathbf{r},t)\tilde{f}_{14}(\mathbf{r}',t')\rangle = \frac{2k_BT_0}{A\tau_0}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \quad (46)$$

The correlators (46), appear to present some problems, in that they diverge when one takes the Newtonian limit $\tau_i \to 0$. But this is an illusion: one has to expose the τ_i dependence explicitly in the dynamical equation itself, and not just in the stochastic term, in order to understand this limit. We will discuss this further in the next section, where we will also show that by reformulating the prescription derived in this section, we can make direct contact with the fluctuating dynamics of LIT discussed in Sec. III.

V. AN ALTERNATIVE FORMULATION

In the previous section, we deduced the FDT for hyperbolic phenomena. The equations for a_1, $a_{\mu+1}$, and a_5, had no stochastic terms whereas those for $\mathring{a}_{\mu\nu}$, $a_{\mu+10}$, and a_{14} did; their two-point correlation functions are given by Eq. (46). In this section, we will explore this structure in more detail and show that an alternative formulation exists, in which the $\mathring{a}_{\mu\nu}$, $a_{\mu+10}$, and a_{14} are eliminated and thus, just as in LIT, only equations for a_1, $a_{\mu+1}$, and a_5 remain. Also, again just as in LIT, the equations for $a_{\mu+1}$ and a_5 have stochastic terms, but the equation for a_1 does not. However, unlike LIT, these stochastic terms are not δ-function correlated in time. In an attempt to make the demonstration as clear as possible, let us focus only on the pair of equations for a_5 and $a_{\mu+10}$; a similar discussion holds for the sets $a_{\mu+1}$ and $\{\mathring{a}_{\mu\nu},a_{14}\}$.

From Eq. (A16) in the Appendix and the fact that $\tilde{f}_5$ vanishes, the equation for $a_5(\mathbf{r},t)$ reads

$$\frac{\partial a_5}{\partial t} + Y_1\frac{\partial a_{\mu+1}}{\partial x_\mu} + \tau_1^{-1/2}Y_2\frac{\partial a_{\mu+10}}{\partial x_\mu} = 0, \quad (47)$$

and from Eq. (A18) the equation for $a_{\mu+10}(\mathbf{r},t)$ reads

$$\frac{\partial a_{\mu+10}}{\partial t} + \tau_1^{-1}a_{\mu+10} = -\tau_1^{-1/2}Y_2\frac{\partial a_5}{\partial x_\mu} + \tilde{f}_{\mu+10}. \quad (48)$$

Here Y_1 and Y_2 are two τ_i-independent constants given by

$$Y_1 = \frac{B}{\rho_0}\left(\frac{T_0}{C}\right)^{1/2} \quad \text{and} \quad Y_2 = \left(\frac{\lambda}{\rho_0 C}\right)^{1/2}. \quad (49)$$

We can immediately integrate Eq. (48) to obtain

$$a_{\mu+10}(\mathbf{r},t) = -\tau_1^{-1/2}Y_2\int_{-\infty}^{t}dt'\,\exp\{-(t-t')/\tau_1\}\frac{\partial a_5(\mathbf{r},t')}{\partial x_\mu} - \left(\frac{\tau_1}{\lambda AT_0}\right)^{1/2}\tilde{G}_\mu(\mathbf{r},t), \quad (50)$$

where the new stochastic quantity, $\tilde{G}_\mu$, is defined by

$$\tilde{G}_\mu(\mathbf{r},t) \equiv -\left(\frac{\lambda AT_0}{\tau_1}\right)^{1/2}\int_{-\infty}^{t}dt'\,\exp\{-(t-t')/\tau_1\} \times \tilde{f}_{\mu+10}(\mathbf{r},t'). \quad (51)$$

In Eq. (50) we have used $\lim_{t\to-\infty}e^{t/\tau_1}a_{\mu+10}(\mathbf{r},t)=0$. That is, the system is aged, and the initial condition has been set in the infinitely distant past.

We now, as indicated in the introduction to this section, eliminate $a_{\mu+10}$ in Eq. (47) by using Eq. (50). Since the combination $\lambda\tau_1^{-1}$ multiplied by the exponential function in Eq. (50) [and in Eq. (51)] will keep on appearing, let us define the function

$$\lambda(t) \equiv \begin{cases} \lambda\tau_1^{-1}\exp\{-t/\tau_1\}, & \text{if} \quad t \geqslant 0 \\ 0, & \text{if} \quad t<0. \end{cases} \quad (52)$$

Then Eq. (47) becomes

$$\frac{\partial a_5(\mathbf{r},t)}{\partial t}-\frac{1}{\rho_0 C}\int_{-\infty}^{\infty}dt'\,\lambda(t-t')\frac{\partial^2 a_5(\mathbf{r},t')}{\partial x_\mu\,\partial x_\mu}+Y_1\frac{\partial a_{\mu+1}(\mathbf{r},t)}{\partial x_\mu}$$
$$=(\rho_0 T_0 AC)^{-1/2}\frac{\partial \widetilde{G}_\mu(\mathbf{r},t)}{\partial x_\mu}. \tag{53}$$

Let us first note the presence of the memory kernel $\lambda(t-t')$. In the limit $\tau_1\to 0,\lambda(t)\to\lambda\,\delta(t)$, and therefore the left-hand side of Eq. (53) becomes

$$\frac{\partial a_5}{\partial t}-\frac{\lambda}{\rho_0 C}\frac{\partial^2 a_5}{\partial x_\mu\,\partial x_\mu}+Y_1\frac{\partial a_{\mu+1}}{\partial x_\mu}, \tag{54}$$

in agreement with the results of Ref. [6]. For systems that obey the FDT, we would therefore expect that the stochastic term on the right-hand side of Eq. (53) is also not δ-function correlated, but has a temporal correlation related to the function $\lambda(t)$ [19,20]. To investigate this point further, we notice that the stochastic term involves the derivative of $\widetilde{G}_\mu$ and so it is very natural to make the analogous transformation to Eq. (35) and to define

$$\widetilde{F}_5(\mathbf{r},t)=(\rho_0 T_0 AC)^{-1/2}\frac{\partial}{\partial x_\mu}\widetilde{G}_\mu(\mathbf{r},t), \tag{55}$$

so that the right-hand side of Eq. (53) is simply $\widetilde{F}_5$. From Eq. (51) it follows that since $\widetilde{f}_{\mu+10}$ is Gaussianly distributed with zero mean, then $\widetilde{G}_\mu$ is also Gaussianly distributed with zero mean. Moreover, a short calculation using the second expression in Eq. (46) yields

$$\langle\widetilde{G}_\mu(\mathbf{r},t)\widetilde{G}_\nu(\mathbf{r}',t')\rangle=\frac{k_B T_0^2\lambda}{\tau_1}\,\delta_{\mu\nu}\,\exp\{-|t-t'|/\tau_1\}\,\delta(\mathbf{r}-\mathbf{r}')$$
$$=k_B T_0^2\delta_{\mu\nu}\lambda(|t-t'|)\,\delta(\mathbf{r}-\mathbf{r}'). \tag{56}$$

This is exactly as we would expect on general grounds for a theory that obeys a FDT. The correlation function of the stochastic terms involves the same function as appears in the memory kernel, but with the argument of the function being the modulus of the argument of the memory kernel [19,20].

Let us pause to summarize what has so far been demonstrated in this section. We have shown that there are two alternative formulations of the equation for a_5 in the fluctuating dynamics of EIT:

(a) A single equation (53) may be given. This equation has a memory term and a stochastic term (55) that is exponentially correlated in time and whose correlation function is given by Eq. (56).

(b) Two Eqs. (47) and (48) may be given. These have no memory term, and only one has a stochastic term that is δ-function correlated, with a correlation function that is given by the appropriate element of Q_{bc}.

This situation is familiar in the theory of non-Markovian stochastic processes [21]. In certain cases the non-Markovian process may be made Markovian by extending the space of variables. The case of exponentially-correlated noise is in many ways the simplest generalization of the δ-function (i.e., Markovian) case and this process can be made Markovian by enlarging the space from a_5 to $\{a_5,a_{\mu+10}\}$. In this case, the equation for the original variable has no stochastic term and the equation for the newly introduced variable does have a stochastic term, which is now, however, δ-function correlated. Both formulations have their advantages; the single equation is useful in that the generalization from LIT is somewhat more obvious, whereas the two separate equations are useful when one wishes to make use of the theory of Markovian processes. The fact that fluctuations in EIT involve non-Markovian processes was noticed some time ago [22], although there has been some dispute over some of the statements that appear in this paper [23]. Our treatment differs in that we concentrate on the explicit formulas, and stresses the role of the FDT.

We have already remarked on several occasions on the limit $\tau_1\to 0$ of the single-equation formulation. At the end of Sec. IV we commented that the Markovian formulation appeared to have some problems when this limit was taken. To show that these problems are not real we need to recall that while a_5 is a scaled version of the physical field T_1 for which the scaling does not involve τ_1, $a_{\mu+10}$ is a scaled version of q_μ for which the scaling does involve τ_1. Therefore, while all the τ_1 dependence in Eq. (53) is manifest, and we may simply take the limit by considering the limit of the λ-function (52), in the two equations (47) and (48) we need to make the τ_1 dependence explicit by defining new quantities:

$$a_{\mu+10}=\tau_1^{1/2}\mathcal{A}_{\mu+10},\quad \widetilde{f}_{\mu+10}=\tau_1^{-1/2}\widetilde{\xi}_{\mu+10}. \tag{57}$$

The first transformation is dictated by Eq. (41): the field $\mathcal{A}_{\mu+10}$ has a finite limit as $\tau_1\to 0$. The second transformation is dictated by Eq. (46): the correlation function of the stochastic quantity $\widetilde{\xi}_{\mu+10}$ also has a finite limit as $\tau_1\to 0$. Making the substitutions (57) in Eqs. (47) and (48) gives

$$\frac{\partial a_5}{\partial t}+Y_1\frac{\partial a_{\mu+1}}{\partial x_\mu}+Y_2\frac{\partial\mathcal{A}_{\mu+10}}{\partial x_\mu}=0 \tag{58}$$

and

$$\tau_1\frac{\partial\mathcal{A}_{\mu+10}}{\partial t}+\mathcal{A}_{\mu+10}=-Y_2\frac{\partial a_5}{\partial x_\mu}+\widetilde{\xi}_{\mu+10}. \tag{59}$$

All the τ_1 dependence is now manifest. Taking the $\tau_1\to 0$ limit in Eq. (59) and substituting for $\mathcal{A}_{\mu+10}$ in Eq. (58) gives

$$\frac{\partial a_5}{\partial t}+Y_1\frac{\partial a_{\mu+1}}{\partial x_\mu}-Y_2^2\frac{\partial^2 a_5}{\partial x_\mu\,\partial x_\mu}=-Y_2\frac{\partial\widetilde{\xi}_{\mu+10}}{\partial x_\mu}, \tag{60}$$

where

$$\langle\widetilde{\xi}_{\mu+10}(\mathbf{r},t)\widetilde{\xi}_{\nu+10}(\mathbf{r}',t')\rangle=\frac{2k_B T_0}{A}\,\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{61}$$

This is in agreement with Eq. (54), which was obtained as the limit $\tau_1\to 0$ of the nonlocal equation (53), and the stochastic term of LIT defined by Eqs. (35) and (37).

Another interesting interpretation may be found by writing Eq. (53) in frequency space, that is, taking its Fourier transform in the time variable. Doing this one finds

$$-i\omega a_5(\mathbf{r},\omega)-\frac{\lambda(\omega)}{\rho_0 C}\frac{\partial^2 a_5(\mathbf{r},\omega)}{\partial x_\mu\,\partial x_\mu}+Y_1\frac{\partial a_{\mu+1}(\mathbf{r},\omega)}{\partial x_\mu}=\tilde{F}_5(\mathbf{r},\omega). \tag{62}$$

Here $\lambda(\omega)$ is the Fourier transform of the function $\lambda(t)$ defined by Eq. (52):

$$\lambda(\omega)=\int_{-\infty}^{\infty}dt\; e^{i\omega t}\lambda(t)=\frac{\lambda}{1-i\omega\tau_1}. \tag{63}$$

Since

$$\int_{-\infty}^{\infty}dt\; e^{i\omega t}\lambda(|t|)=2\,\mathrm{Re}\,\lambda(\omega)=\frac{2\lambda}{1+\omega^2\tau_1^2}, \tag{64}$$

we may write the correlation function (56) of the $\tilde{G}_\mu(\mathbf{r},\omega)$ in frequency space as

$$\langle\tilde{G}_\mu(\mathbf{r},\omega)\tilde{G}_\nu^*(\mathbf{r}',\omega)\rangle=2(2\pi)k_BT_0^2\frac{\lambda\,\delta_{\mu\nu}}{[1+\omega^2\tau_1^2]}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{65}$$

Notice that the structure in which $\lambda(t)$ appears in the memory kernel and $\lambda(|t|)$ in the correlation function of the stochastic term—a structure that follows from the FDT of Sec. IV—means that where λ appeared in the deterministic part of the equations of LIT, $\lambda(\omega)$ now appears, and where λ appeared in the correlation function of the stochastic terms in LIT, $\mathrm{Re}\,\lambda(\omega)$ now appears.

It is now straightforward to extend this procedure to the equation for $a_{\mu+1}$. By constructing an analogous set of arguments to those described above, we find that the generalization of the LIT equation for $a_{\mu+1}$ is simply found by replacing the transport coefficients μ and ζ by

$$\mu(\omega)=\frac{\mu}{1-i\omega\tau_2};\quad \zeta(\omega)=\frac{\zeta}{1-i\omega\tau_0}, \tag{66}$$

respectively. That is,

$$-i\omega a_{\rho+1}(\mathbf{r},\omega)-\frac{1}{\rho_0}\{2\mu(\omega)X_{\rho\mu\sigma\nu}+\zeta(\omega)\delta_{\rho\mu}\delta_{\sigma\nu}\}\times\frac{\partial^2 a_{\sigma+1}(\mathbf{r},\omega)}{\partial x_\mu\,\partial x_\nu}+A^{1/2}\frac{\partial a_1(\mathbf{r},\omega)}{\partial x_\rho}+Y_1\frac{\partial a_5(\mathbf{r},\omega)}{\partial x_\rho}=\tilde{F}_{\rho+1}(\mathbf{r},\omega). \tag{67}$$

The analogous equation to Eq. (34) is

$$\tilde{F}_{\mu+1}(\mathbf{r},\omega)=(\rho_0A)^{-1/2}\frac{\partial}{\partial x_\nu}\tilde{S}_{\mu\nu}(\mathbf{r},\omega), \tag{68}$$

with

$$\langle\tilde{S}_{\mu\nu}(\mathbf{r},\omega)\tilde{S}_{\rho\sigma}^*(\mathbf{r}',\omega)\rangle=2(2\pi)k_BT_0\{2\,\mathrm{Re}\,\mu(\omega)X_{\mu\nu\rho\sigma}+\mathrm{Re}\,\zeta(\omega)\,\delta_{\mu\nu}\delta_{\rho\sigma}\}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{69}$$

This agrees with [24] when $\tau_0=\tau_2$.

Finally, from the Appendix we see that the equation for a_1 does not involve any of the variables $\mathring{a}_{\mu\nu}$, $a_{\mu+10}$, or a_{14}. Therefore, it takes the same form as in LIT:

$$\frac{\partial a_1(\mathbf{r},t)}{\partial t}+A^{1/2}\frac{\partial a_{\mu+1}(\mathbf{r},t)}{\partial x_\mu}=0. \tag{70}$$

VI. CONCLUSIONS

In this paper we have explored several aspects of fluctuations in macroscopic systems where the thermodynamic inertia is not negligible. We began by reviewing the situation in LIT where the inertia is negligible, and showed how the statistics of the fluctuations can be completely determined by features of the macroscopic theory. While we followed Fox and Uhlenbeck [6] for the first part of the derivation, we adopted a more direct approach than they did by using the Einstein formula $P_S\sim\exp\delta^2S/2k_B$, rather than dS/dt, to determine E_{bc}^{jk}. This indirect determination of the matrix E, which is a feature of the Fox-Uhlenbeck treatment, is somewhat unsatisfactory, but does circumvent the question of whether velocity fluctuations should be included in δ^2S. Fortunately, this issue has been clarified since the publication of the paper by Fox and Uhlenbeck, and a cleaner derivation of the FDT in this case is now possible.

It is natural to try to extend these ideas to EIT: linearizing the theory about the equilibrium state and adding fluctuations to each of the equations for $v,v_\mu,T,\mathring{\tau}_{\mu\nu},q_\mu,\tau$. In order to be able to use the Einstein formula to determine E_{bc}^{jk} it is necessary, as in the LIT case, to assume that velocity is a thermodynamic variable in the theory. It seems that this takes us outside EIT as usually defined [2], although occasionally it has been explicitly included [25]. In any case, we find it difficult to understand how a consistent theory of fluctuations can be formulated unless velocity fluctuations are included. For if they are not included, some of the diagonal entries of the E matrix are zero and so the inverse does not exist (within the linear theory) and the FDT (7) cannot be invoked. When the velocity terms are included one finds that not only does the equation for a_1 have no stochastic term (which we would expect, since it does not contain any dissipative constants), but the equations for $a_{\mu+1}$ and a_5 have none either. On the other hand, stochastic terms do appear in the equations for $\mathring{a}_{\mu\nu}$, $a_{\mu+10}$, and a_{14}, and these, unlike those in LIT, have correlation functions that do not involve spatial derivatives of the δ function.

This structure is reminiscent of Langevin equations where the noise is not white, but exponentially correlated. In these cases it is possible to increase the number of variables in such a way that the equations for the original variables now have no noise, and those for the newly introduced variables have noise terms that are white, i.e., δ-function correlated. In the case of interest to us in this paper, fluctuating EIT corresponds to generalizing fluctuating LIT by extending the space of variables from the original five $\{a_1, a_{\mu+1}, a_5\}$ by adding the extra nine comprising $\{\mathring{a}_{\mu\nu}, a_{\mu+10}, a_{14}\}$ to make 14 in all. In this larger space the stochastic process that is fluctuating EIT (including the velocity) is Markovian. In the space of the original five variables it is, however, non-Markovian.

The introduction of new variables in a non-Markovian stochastic process in order to render it Markovian, is often a very useful device, since the theory of Markovian processes is usually so much easier. However, the non-Markovian representation has some advantages, the principal one being that it is easier to interpret physically. In Sec. V we showed instances of this in the present problem. For instance, in the temporal representation, the nature of the $\tau_i \to 0$ limit was much clearer than in the Markovian formulation. In the frequency representation, the picture was even clearer, with the real, frequency-independent transport coefficients of LIT being replaced by their complex, frequency-dependent counterparts. Other than this, the deterministic parts of the equations were unchanged. The corresponding change to the correlation function of the stochastic terms was a modification whereby the LIT transport coefficients were replaced by the real parts of the frequency-dependent ones.

We believe that the simple and intuitively appealing structure of fluctuating EIT discussed here will prove useful in the many future applications of the theory.

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APPENDIX

In this appendix we discuss the thermodynamic formulas required in order to prove some of the results that appear in the main text and we also linearize Eqs. (8)–(10) and (38)–(40) about the equilibrium state of the fluid they describe.

We begin by recalling some results from the theory of classical equilibrium thermodynamics. These are required since both the governing equation (10) and the probability distribution (25), are given in terms of the internal energy per unit mass, $u(\mathbf{r},t)$, and we wish to work with the temperature field $T(\mathbf{r},t)$, which has a more immediate physical interpretation. We begin from the relation (going over to the discrete notation for the spatial variable)

$$u_1^j = \left(\frac{\partial u}{\partial v}\right)_T v_1^j + \left(\frac{\partial u}{\partial T}\right)_v T_1^j, \tag{A1}$$

where, following the notation established in the main part of the text, all partial derivatives with subscripts outside of the brackets are evaluated in the equilibrium state. We now use the the standard thermodynamic relation

$$\left(\frac{\partial u}{\partial v}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_v - p, \tag{A2}$$

in Eq. (A1) and obtain

$$u_1^j = [BT_0 - p_0] v_1^j + C T_1^j, \tag{A3}$$

where

$$B = \left(\frac{\partial p}{\partial T}\right)_v, \tag{A4}$$

and C is given by Eq. (14).

We now list the other thermodynamic relations we need to use in derivations in the main text. First, to derive the linearized form of Eq. (9) we use

$$\left(\frac{\partial p}{\partial x_\mu}\right) = A\left(\frac{\partial \rho}{\partial x_\mu}\right) + B\left(\frac{\partial T}{\partial x_\mu}\right), \tag{A5}$$

where A is given by Eq. (14) and B by Eq. (A4).

To prove the relation (26), we first use Eq. (A3), and then the two relations

$$\left(\frac{\partial\{T^{-1}\}}{\partial v}\right)_u + \left(\frac{\partial\{T^{-1}\}}{\partial u}\right)_v [BT_0 - p_0] = 0 \tag{A6}$$

$$\text{and} \quad \left(\frac{\partial p}{\partial \rho}\right)_u = \left(\frac{\partial p}{\partial \rho}\right)_T + \left(\frac{\partial T}{\partial \rho}\right)_u \left(\frac{\partial p}{\partial T}\right)_\rho. \tag{A7}$$

We now move on to the derivation of the linearized equations. Expanding about the equilibrium state $(v, v_\mu, T, \mathring{\tau}_{\mu\nu}, q_\mu, \tau) = (v_0, 0, T_0, 0, 0, 0)$, and denoting volume and temperature fluctuations by v_1 and T_1, respectively, we obtain

$$\rho_0 \frac{\partial v_1}{\partial t} = \frac{\partial (v_1)_\mu}{\partial x_\mu}, \tag{A8}$$

$$\rho_0 \frac{\partial (v_1)_\mu}{\partial t} - \rho_0^2 A \frac{\partial v_1}{\partial x_\mu} + B\frac{\partial T_1}{\partial x_\mu} = -\frac{\partial}{\partial x_\nu}(\tau \delta_{\mu\nu} + \mathring{\tau}_{\mu\nu}), \tag{A9}$$

$$\rho_0 C \frac{\partial T_1}{\partial t} + T_0 B \frac{\partial (v_1)_\mu}{\partial x_\mu} = -\frac{\partial q_\mu}{\partial x_\mu}, \tag{A10}$$

$$\tau_2 \frac{\partial \mathring{\tau}_{\mu\nu}}{\partial t} + \mathring{\tau}_{\mu\nu} = -2\mu X_{\mu\nu\rho\sigma} \frac{\partial (v_1)_\rho}{\partial x_\sigma}, \tag{A11}$$

$$\tau_1 \frac{\partial q_\mu}{\partial t} + q_\mu = -\lambda \frac{\partial T_1}{\partial x_\mu}, \tag{A12}$$

$$\tau_0 \frac{\partial \tau}{\partial t} + \tau = -\zeta \frac{\partial (v_1)_\mu}{\partial x_\mu}. \tag{A13}$$

We now rewrite the linearized equations (A8)–(A13) in terms of the new physical fields (13) and (41). We find

$$\partial_t a_1 + (A)^{1/2} \partial_\mu a_{\mu+1} = 0, \tag{A14}$$

$$\partial_t a_{\mu+1} + (A)^{1/2} \partial_\mu a_1 + \frac{B}{\rho_0}\left(\frac{T_0}{C}\right)^{1/2} \partial_\mu a_5 + \left(\frac{\mu}{\rho_0 \tau_2}\right)^{1/2} \partial_\nu \mathring{a}_{\mu\nu} + \left(\frac{\zeta}{\rho_0 \tau_0}\right)^{1/2} \partial_\mu a_{14} = 0, \tag{A15}$$

$$\partial_t a_5 + \frac{B}{\rho_0}\left(\frac{T_0}{C}\right)^{1/2} \partial_\mu a_{\mu+1} + \left(\frac{\lambda}{\tau_1 \rho_0 C}\right)^{1/2} \partial_\mu a_{\mu+10} = 0, \tag{A16}$$

$$\partial_t \mathring{a}_{\mu\nu} + \tau_2^{-1} \mathring{a}_{\mu\nu} + 2\left(\frac{\mu}{\rho_0 \tau_2}\right)^{1/2} X_{\mu\nu\rho\sigma} \partial_\rho a_{\sigma+1} = 0, \tag{A17}$$

$$\partial_t a_{\mu+10} + \tau_1^{-1} a_{\mu+10} + \left(\frac{\lambda}{\tau_1 \rho_0 C}\right)^{1/2} \partial_\mu a_5 = 0, \tag{A18}$$

$$\partial_t a_{14} + \tau_0^{-1} a_{14} + \left(\frac{\zeta}{\tau_0 \rho_0}\right)^{1/2} \partial_\mu a_{\mu+1} = 0, \tag{A19}$$

where with obvious notation ∂_μ means $\partial/\partial x_\mu$ and ∂_t means $\partial/\partial t$.

In order to make contact with the Langevin form (2), we need to make the spatial dependence more explicit. Therefore, we wish to write Eq. (A14), for example, in the form

$$\frac{\partial a_1(\mathbf{r},t)}{\partial t} + \int d\mathbf{r}' G_{1b}(\mathbf{r},\mathbf{r}') a_b(\mathbf{r}',t) = 0, \quad b = 1,\ldots,14. \tag{A20}$$

Comparison with Eq. (A14) shows that the only $G_{1b}(\mathbf{r},\mathbf{r}')$ that are nonzero are those with $b = 2,3,4$. In this case

$$G_{1(\mu+1)}(\mathbf{r},\mathbf{r}') = (A)^{1/2} \partial_\mu \delta(\mathbf{r}-\mathbf{r}'). \tag{A21}$$

In a similar way we can determine all of the other $G_{bc}(\mathbf{r},\mathbf{r}')$. From Eqs. (A14)–(A19), it is immediately apparent that

$$G_{bc}(\mathbf{r},\mathbf{r}') = G_{cb}(\mathbf{r},\mathbf{r}'), \tag{A22}$$

or in the discrete form introduced in Sec. I, $G^{jk}_{bc} = G^{jk}_{cb}$. In fact, in terms of the symmetric and antisymmetric forms defined by Eqs. (19) and (20), we further note that $A^{jk}_{bc} = A^{jk}_{cb}$ and $S^{jk}_{bc} = S^{jk}_{cb}$. In continuous notation this reads

$$A_{bc}(\mathbf{r},\mathbf{r}') = A_{cb}(\mathbf{r},\mathbf{r}'); \quad S_{bc}(\mathbf{r},\mathbf{r}') = S_{cb}(\mathbf{r},\mathbf{r}'). \tag{A23}$$

Therefore, we need to only list the A_{bc} for $b < c$ and the S_{bc} for $b \leqslant c$; Eq. (A23) gives the others once these are known. Notice also that Eqs. (20) and (A23) taken together mean that $A_{bc}(\mathbf{r},\mathbf{r}')$ and $S_{bc}(\mathbf{r},\mathbf{r}')$ are antisymmetric and symmetric, respectively, under the interchange $\mathbf{r} \leftrightarrow \mathbf{r}'$.

The explicit forms are as follows. The only nonzero S_{bc} are

$$\mathring{S}_{\mu\nu,\rho\sigma}(\mathbf{r},\mathbf{r}') = \tau_2^{-1} X_{\mu\nu\rho\sigma} \delta(\mathbf{r}-\mathbf{r}'),$$

$$S_{(\mu+10)(\nu+10)}(\mathbf{r},\mathbf{r}') = \tau_1^{-1} \delta_{\mu\nu} \delta(\mathbf{r}-\mathbf{r}'),$$

$$S_{1414}(\mathbf{r},\mathbf{r}') = \tau_0^{-1} \delta(\mathbf{r}-\mathbf{r}'). \tag{A24}$$

The only nonzero A_{bc} with $b < c$ are

$$A_{1(\mu+1)}(\mathbf{r},\mathbf{r}') = A^{1/2} \frac{\partial}{\partial x_\mu} \delta(\mathbf{r}-\mathbf{r}'),$$

$$A_{(\mu+1)5}(\mathbf{r},\mathbf{r}') = \frac{B}{\rho_0}\left(\frac{T_0}{C}\right)^{1/2} \frac{\partial}{\partial x_\mu} \delta(\mathbf{r}-\mathbf{r}'),$$

$$A_{(\rho+1),\mu\nu}(\mathbf{r},\mathbf{r}') = \left(\frac{\mu}{\rho_0 \tau_2}\right)^{1/2} X_{\mu\nu\rho\sigma} \frac{\partial}{\partial x_\sigma} \delta(\mathbf{r}-\mathbf{r}'),$$

$$A_{(\mu+1)14}(\mathbf{r},\mathbf{r}') = \left(\frac{\zeta}{\rho_0 \tau_0}\right)^{1/2} \frac{\partial}{\partial x_\mu} \delta(\mathbf{r}-\mathbf{r}'),$$

$$A_{5(\mu+10)}(\mathbf{r},\mathbf{r}') = \left(\frac{\lambda}{\tau_1 \rho_0 C}\right)^{1/2} \frac{\partial}{\partial x_\mu} \delta(\mathbf{r}-\mathbf{r}'). \tag{A25}$$

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